Characterizations for the commutator of parabolic nonsingular integral operator on parabolic generalized Orlicz-Morrey spaces

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Abstract

We show continuity in parabolic generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}$ of commutator of parabolic nonsingular integral operators. We shall give necessary and sufficient conditions for the boundedness of the commutator of parabolic nonsingular integral operator on $M^{\Phi,\varphi}$ spaces with BMO functions.

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1 Introduction and main results

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [4, 5] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A celebrated result of Coifman, Rochberg and Weiss [8] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in the study of regularity of solutions of elliptic partial differential equations of second order (see, for example, [6, 7, 10, 26, 27]).

The classical Morrey spaces were introduced by Morrey [35] to study the local behavior of solutions to second-order elliptic partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [16, 34, 36] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [17, 18, 45]). Later, Guliyev [18] defined the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with normalized norm

$$\|f\|_{M^{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-1/p} \, \|f\|_{L^p(B(x,r))},$$

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

The Orlicz space were first introduced by Orlicz in [42, 43] as generalizations of Lebesgue spaces $L^p(\mathbb{R}^n)$. Since then, the theory of Orlicz spaces themselves has been well developed and the spaces

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have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis.

In [11], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [37] and [44]. In words of [24], our generalized Orlicz-Morrey space is the third kind and the ones in [37] and [44] are the first kind and the second kind, respectively. According to the examples in [15], one can say that the generalized Orlicz-Morrey spaces of the first kind and the second kind are different and that second kind and third kind are different. However, we do not know the relation between the first and the second kind.

Note that, Orlicz-Morrey spaces unify Orlicz and generalized Morrey spaces. We extend some results on generalized Morrey space in the papers [13, 18, 20, 21, 25, 29] to the case of Orlicz-Morrey space in [11, 14, 22, 23, 24].

As based on the results of [18, 20], the following conditions were introduced in [11] (see, also [22]) for the boundedness of the singular integral operators on $M^{\Phi,\varphi}(\mathbb{R}^n)$,

$$\int_r^\infty \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)}\Big) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x,r),$$

where C does not depend on x and r.

Consider the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$. For $x=(x',t)\in\mathbb{R}^{n+1}_+$, $x=(x'',x_n,t)\in\mathbb{D}^{n+1}_+ = \mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R}_+$, $\mathbb{D}^{n+1}_- = \mathbb{R}^{n-1}\times\mathbb{R}_-\times\mathbb{R}_+$. In the following, besides the standard parabolic metric $\varrho(x)=\max(|x'|,|t|^{1/2})$ we use the equivalent one $\rho(x)=\left(\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [9]. The induced by it topology consists of ellipsoids (parabolic balls)

$$\mathcal{E}_r(x)=\left\{y\in\mathbb{R}^{n+1}:\ \frac{|x'-y'|^2}{r^2}+\frac{|t-\tau|^2}{r^4}<1\right\},\ |\mathcal{E}_r|=Cr^{n+2}.$$

It is easy to see that $\mathcal{E}_1(x)$ and $\mathbb{S}^n$ are the unit ball and the unit sphere, respectively, with respect to the both metrics and $\rho(x)$. On the other hand, the equivalence between the both parabolic metrics $\varrho(x)$ and $\rho(x)$ follows by the inclusion: for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ with measure comparable with r^{n+2} such that $\underline{\mathcal{C}}\subset\mathcal{E}_r\subset\overline{\mathcal{C}}$. In what follows all estimate obtained over ellipsoids hold true also over parabolic cylinders and we shall use this property without explicit references.

Let $\widetilde{x}=(x'',-x_n,t)$ be the "reflected point". The parabolic nonsingular integral operator $\mathcal{R}$ is defined by (see [2])

$$\mathcal{R}f(x)=\int_{\mathbb{D}^{n+1}_+}\frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}\,dy. \tag{1.1}$$

The commutators generated by $b\in L^1_{\rm loc}(\mathbb{D}^{n+1}_+)$ and the operator $\mathcal{R}$ are defined by

$$[b,\mathcal{R}]f(x)=\int_{\mathbb{D}^{n+1}_+}\frac{b(x)-b(y)}{\rho(\widetilde{x}-y)^{n+2}}\,f(y)\,dy.$$

The operator $|b,\mathcal{R}|$ is defined by

$$|b,\mathcal{R}|f(x)=\int_{\mathbb{D}^{n+1}_+}\frac{|b(x)-b(y)|}{\rho(\widetilde{x}-y)^{n+2}}\,f(y)\,dy.$$

The operator $\mathcal{R}$ and its commutator appear in [2] in connection with boundary estimates for solutions to parabolic equations.

In [40, 41] the author was study the boundedness of the parabolic nonsingular integral operator $\mathcal{R}$ on Orlicz and generalized Orlicz-Morrey spaces, respectively. Therefore, the purpose of this paper is mainly to study the boundedness of the commutator of parabolic nonsingular integral operator $[b, \mathcal{R}]$ on parabolic generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ with BMO functions.

Therefore, the purpose of this paper is mainly to study the boundedness of the commutators of parabolic nonsingular operator $[b, \mathcal{R}]$ on parabolic generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{D}^{n+1}_+)$. We give necessary and sufficient conditions for the boundedness of the commutators of parabolic nonsingular operator $|b, \mathcal{R}|$ on parabolic generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}(\mathbb{D}^{n+1}_+)$, respectively.

A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all decreasing functions $\varphi : (0, \infty) \to (0, \infty)$ such that $t \in (0, \infty) \mapsto \Phi^{-1}(t^{-n-2})\varphi(t)^{-1}$ is almost decreasing.

The following results are the fundamental theorems in this paper:

Theorem 1.1. Let $b \in BMO(\mathbb{D}^{n+1}_+)$, Φ be a Young function with $\Phi \in \Delta_2$ and $\varphi_1, \varphi_2 \in \Omega_\Phi$.

1. If $\Phi \in \nabla_2$, then the condition

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x, s)}{\Phi^{-1}\big(s^{-n-2}\big)}\Big) \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t} \le C\, \varphi_2(x, r), \tag{1.2}$$

where C does not depend on x and r, is sufficient for the boundedness of $|b, \mathcal{R}|$ from $M^{\Phi,\varphi_1}(\mathbb{R}^{n+1}_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^{n+1}_+)$.

2. If $\varphi_1 \in \mathcal{G}_\Phi$, then the condition

$$\varphi_1(x, r) \le C\varphi_2(x, r), \tag{1.3}$$

where C does not depend on x and r, is necessary for the boundedness of $|b, \mathcal{R}|$ from $M^{\Phi,\varphi_1}(\mathbb{R}^{n+1}_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^{n+1}_+)$.

3. If $\Phi \in \nabla_2$ and $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition

$$\int_t^\infty \varphi_1(r) \frac{dr}{r} \le C\varphi_1(t), \tag{1.4}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (1.3) is necessary and sufficient for the boundedness of $|b, \mathcal{R}|$ from $M^{\Phi,\varphi_1}(\mathbb{R}^{n+1}_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^{n+1}_+)$.

If we take $\Phi(t) = t^p$, $p \in [1, \infty)$ at Theorem 1.1 we get the following new result for generalized Morrey spaces.

Corollary 1.2. Let $p \in [1,\infty)$, $b \in BMO(\mathbb{D}_+^{n+1})$ and $\varphi_1, \varphi_2 \in \Omega_p \equiv \Omega_{t^p}$.

1. If $1 < p < \infty$, then the condition

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{\frac{n+2}{p}}}{t^{\frac{n+2}{p}+1}} dt \le C\varphi_2(r), \tag{1.5}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $|b, \mathcal{R}|$ from $M^{p,\varphi_1}(\mathbb{D}_+^{n+1})$ to $M^{p,\varphi_2}(\mathbb{D}_+^{n+1})$.

2. If $\varphi_1 \in \mathcal{G}_p$, then the condition (1.3) is necessary for the boundedness of $|b, \mathcal{R}|$ from $M^{p,\varphi_1}(\mathbb{D}_+^{n+1})$ to $M^{p,\varphi_2}(\mathbb{D}_+^{n+1})$.

3. If $1 < p < \infty$ and $\varphi_1 \in \mathcal{G}_p$ satisfies the regularity condition (1.4), then the condition (1.3) is necessary and sufficient for the boundedness of $|b, \mathcal{R}|$ from $M^{p,\varphi_1}(\mathbb{D}_+^{n+1})$ to $M^{p,\varphi_2}(\mathbb{D}_+^{n+1})$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Definitions and preliminary results

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(r) := \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) g(t) w(t) dt, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [19] (see also [28]).

Theorem 2.1. [19] Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{2.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int_r^\infty \big(1 + \ln\frac{t}{r}\big) \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{2.2}$$

Moreover, the value $C = B$ is the best constant for (2.1).

Remark 2.2. In (2.1) and (2.2) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

2.1 On Young Functions and Orlicz Spaces

We recall the definition of Young functions.

Definition 2.3. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \le r < \infty.$$

It is well known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{2.3}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

Definition 2.4. (Orlicz Space). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{D}_+^{n+1}) = \left\{ f \in L^1_{\text{loc}}(\mathbb{D}_+^{n+1}) : \int_{\mathbb{D}_+^{n+1}} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{D}_+^{n+1}) = L^p(\mathbb{D}_+^{n+1})$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{D}_+^{n+1}) = L^{\infty}(\mathbb{D}_+^{n+1})$. The space $L^{\Phi}_{\text{loc}}(\mathbb{D}_+^{n+1})$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L^{\Phi}(\mathbb{D}_+^{n+1})$ for all balls $\mathcal{E} \subset \mathbb{D}_+^{n+1}$.

$L^{\Phi}(\mathbb{D}_+^{n+1})$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{D}_+^{n+1})} = \inf\left\{\lambda > 0 : \int_{\mathbb{D}_+^{n+1}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

We note that

$$\int_{\mathbb{D}_+^{n+1}} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\mathbb{D}_+^{n+1})}}\Big)dx \le 1. \tag{2.4}$$

The weak Orlicz space

$$WL^{\Phi}(\mathbb{D}_+^{n+1}) = \{f \in L^1_{\text{loc}}(\mathbb{D}_+^{n+1}) : \|f\|_{WL^{\Phi}(\mathbb{D}_+^{n+1})} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}(\mathbb{D}_+^{n+1})} = \inf\Big\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big) \ \leq 1\Big\}.$$

The following lemmas are valid.

Lemma 2.5. [1, 33] Let Φ be a Young function and $\mathcal{E}$ a set in $\mathbb{D}_+^{n+1}$ with finite Lebesgue measure. Then

$$\|\chi_{\mathcal{E}}\|_{WL^{\Phi}(\mathbb{D}_+^{n+1})} = \|\chi_{\mathcal{E}}\|_{L^{\Phi}(\mathbb{D}_+^{n+1})} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

Lemma 2.6. For a Young function Φ and for all parabolic balls $\mathcal{E}$ in $\mathbb{D}_+^{n+1}$, the following inequality is valid

$$\|f\|_{L^1(\mathcal{E})} \leq 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\|f\|_{L^{\Phi}(\mathcal{E})}.$$

2.2 Parabolic generalized Orlicz-Morrey Space

Various versions of generalized Orlicz-Morrey spaces were introduced in [37], [44] and [11]. We used the definition of [11] which runs as follows.

We now define parabolic generalized Orlicz-Morrey spaces of the third kind. The *parabolic generalized Orlicz-Morrey space* $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \equiv \sup_{x\in\mathbb{D}_+^{n+1},\ r>0} \frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\ \|f\|_{L^{\Phi}(\mathcal{E}^+(x,r))}$$

is finite, where $\mathcal{E}^+(x,r) = \mathcal{E}(x,r)\cap\mathbb{D}_+^{n+1}$. Also by $WM^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ we denote the *weak parabolic generalized Orlicz-Morrey space of the third kind* of all functions $f \in WL^{\Phi}_{\text{loc}}(\mathbb{D}_+^{n+1})$ for which

$$\|f\|_{WM^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} = \sup_{x\in\mathbb{D}_+^{n+1}, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{E}^+(x,r)|^{-1})\,\|f\|_{WL^{\Phi}(\mathcal{E}^+(x,r))} < \infty,$$

where $WL^{\Phi}(\mathcal{E}^+(x,r))$ denotes the weak L^{Φ}-space of measurable functions f for which

$$\|f\|_{WL^{\Phi}(\mathcal{E}^+(x,r))} \equiv \|f\chi_{\mathcal{E}^+(x,r)}\|_{WL^{\Phi}(\mathbb{D}_+^{n+1})}.$$

Note that $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ covers many classical function spaces.

Example 2.7. Let $1 \leq q \leq p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(\mathbb{D}_+^{n+1})$, the *classical Morrey space* $M^p_q(\mathbb{D}_+^{n+1})$, the generalized Morrey space $M^{\varphi,p}(\mathbb{D}_+^{n+1})$ and the Orlicz space $L^{\Phi}(\mathbb{D}_+^{n+1})$ with norm coincidence:

1. If $\Phi(t) = t^p$ and $\varphi(t) = t^{-\frac{n+2}{p}}$, then $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1}) = L^p(\mathbb{D}_+^{n+1})$ with norm equivalence.

2. If $\Phi(t) = t^q$ and $\varphi(t) = t^{-\frac{n+2}{p}}$, then $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$, which is denoted by $M^p_q(\mathbb{D}_+^{n+1})$, is the parabolic Morrey space.

3. If $\Phi(t) = t^p$, then $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1}) = M^{p,\varphi}(\mathbb{D}_+^{n+1})$ is the parabolic generalized Morrey space which were discussed in [16], see also [18, 34, 36].

4. If $\varphi(t) = \Phi^{-1}(t^{-n-2})$, then $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1}) = L^{\Phi}(\mathbb{D}_+^{n+1})$.

Other definitions of generalized Orlicz-Morrey spaces can be found in [15, 37, 38, 39]. Therefore, our definition of generalized Orlicz-Morrey spaces here is named "third kind".

In the case $\varphi(x,r) = \frac{\Phi^{-1}\left(|\mathcal{E}(x,r)|^{-1}\right)}{\Phi^{-1}\left(|\mathcal{E}(x,r)|^{-\lambda/n}\right)}$, we get the parabolic Orlicz-Morrey space $M^{\Phi,\lambda}(\mathbb{R}^n)$ from parabolic generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$. We refer to [12, Lemmas 2.8 and 2.9] for more information about Orlicz-Morrey spaces.

Lemma 2.8. [12, Lemma 2.12] Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$.

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|\mathcal{E}(x,r)|^{-1})}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.5}$$

then $M^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in \mathbb{R}^n, \tag{2.6}$$

then $M^{\Phi,\varphi}(\mathbb{R}^n) = \Theta$.

Remark 2.9. Let Φ be a Young function. We denote by Ω_Φ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|\mathcal{E}(x,r)|^{-1})}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.8, we always assume that $\varphi \in \Omega_\Phi$.

The following lemma plays a key role in our main results.

Lemma 2.10. [41] Let $\mathcal{E}_0^+ := \mathcal{E}^+(x_0,r_0)$ a parabolic ball in $\mathbb{D}_+^{n+1}$. If $\varphi \in \mathcal{G}_\Phi$, then there exist $C>0$ such that

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{E}_0^+}\|_{M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})} \le \frac{C}{\varphi(r_0)}.$$

Theorem 2.11. Let Φ any Young function, $\varphi_1, \varphi_2 : \mathbb{D}_+^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying (1.2).

i) If $\Phi \in \Delta_2 \bigcap \nabla_2$, then it is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ in $M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$ and

$$\|\mathcal{R}f\|_{M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})} \le C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})}. \tag{2.7}$$

ii) If $\Phi \in \Delta_2$, then it is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ to $WM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$ and

$$\|\mathcal{R}f\|_{M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})} \le C\|f\|_{WM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})}$$

with constants independent of f.

3 Commutator of parabolic nonsingular integrals in the space $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$

For any $x \in \mathbb{D}_+^{n+1}$ define $\widetilde{x} = (x', -x_n)$ and recall that $x^0 = (x', 0)$. Also define $\mathcal{E}_r^+ \equiv \mathcal{E}^+(x^0, r) = \mathcal{E}(x^0, r) \cap \mathbb{D}_+^{n+1}$, $2\mathcal{E}_r^+ = \mathcal{E}^+(x^0, 2r)$.

We recall the definition of the space of $BMO(\mathbb{D}_+^{n+1})$.

Definition 3.1. Suppose that $f \in L^1_{\rm loc}(\mathbb{D}_+^{n+1})$, let

$$\|f\|_* = \sup_{x\in\mathbb{D}_+^{n+1}, r>0} \frac{1}{|\mathcal{E}^+(x,r)|} \int_{\mathcal{E}^+(x,r)} |f(y) - f_{\mathcal{E}^+(x,r)}| dy,$$

where

$$f_{\mathcal{E}^+(x,r)} = \frac{1}{|\mathcal{E}^+(x,r)|} \int_{\mathcal{E}^+(x,r)} f(y) dy.$$

Define

$$BMO(\mathbb{D}_+^{n+1}) = \{f \in L^1_{\rm loc}(\mathbb{D}_+^{n+1}) \ : \ \|f\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{D}_+^{n+1})$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Before proving our theorems, we need the following lemmas.

Lemma 3.2. [30] Let $b \in BMO(\mathbb{R}^n)$. Then, there is a constant $C > 0$ such that

$$\left|b_{\mathcal{E}^+(x,r)} - b_{\mathcal{E}^+(x,t)}\right| \le C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t, \tag{3.1}$$

where C is independent of b, x, r, and t.

Lemma 3.3. [23, 31] Let $f \in BMO(\mathbb{D}_+^{n+1})$ and Φ be a Young function with $\Phi \in \Delta_2$, then

$$\|f\|_* \approx \sup_{x\in\mathbb{D}_+^{n+1}, r>0} \Phi^{-1}\big(|\mathcal{E}^+(x,r)|^{-1}\big) \left\|f(\cdot) - f_{\mathcal{E}^+(x,r)}\right\|_{L^\Phi(\mathcal{E}^+(x,r))}. \tag{3.2}$$

For a function $b \in BMO$ define the commutator $[b, \mathcal{R}]f = b\mathcal{R}f - \mathcal{R}(bf)$. The following result concerning the boundedness of the operator $[b, \mathcal{R}]$ on L^p space is known.

Theorem 3.4. [2] Let $b \in BMO(\mathbb{D}_+^{n+1})$ and $p \in (1, \infty)$. Then the commutator operator $[b, \mathcal{R}]$ is bounded on $L^p(\mathbb{D}_+^{n+1})$.

From this result and [27, Theorem 2.7], we have the following boundedness of $[b, \mathcal{R}]$ on $L^p(\mathbb{D}_+^{n+1})$.

Theorem 3.5. Let Φ be a Young function with $\Phi \in \Delta_2 \bigcap \nabla_2$ and $b \in BMO(\mathbb{D}_+^{n+1})$. Then the commutator operator $[b, \mathcal{R}]$ is bounded on $L^\Phi(\mathbb{D}_+^{n+1})$.

Our aim is to show boundedness of $[b, \mathcal{R}]$ in $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$.

Lemma 3.6. Let Φ be a Young function with $\Phi \in \Delta_2 \bigcap \nabla_2$ and $b \in BMO(\mathbb{D}_+^{n+1})$. Suppose that for all $f \in L^{\Phi}_{\rm loc}(\mathbb{D}_+^{n+1})$ and $r > 0$ holds

$$\int_1^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(\mathcal{E}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n-2}\big) \, \frac{dt}{t} < \infty. \tag{3.3}$$

Then

$$\|[b,\mathcal{R}]f\|_{L^{\Phi}(\mathcal{E}^+(x^0,r))} \le \frac{C}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(\mathcal{E}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n-2}\big) \, \frac{dt}{t}, \tag{3.4}$$

where the constants are independent of x^0, r and f.

Proof. Denote by $\mathcal{E}_r^+ = \mathcal{E}^+(x^0,r)$, $\mathcal{E}_t^+ = \mathcal{E}^+(x^0,t)$ and for any $f \in L^{\Phi}_{\rm loc}(\mathbb{D}_+^{n+1})$ write $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{E}_r^+}$ and $f_2 = f\chi_{(2\mathcal{E}_r^+)^c}$. Because of the Φ-boundedness of the operator $[b,\mathcal{R}]$ (see Theorem 3.5) and $f_1 \in L^{\Phi}(\mathbb{D}_+^{n+1})$ we have

$$\|[b,\mathcal{R}]f_1\|_{L^{\Phi}(\mathcal{E}_r^+)} \le \|[b,\mathcal{R}]f_1\|_{L^{\Phi}(\mathbb{D}_+^{n+1})} \lesssim \|b\|_* \|f_1\|_{L^{\Phi}(\mathbb{D}_+^{n+1})} = \|b\|_* \|f\|_{L^{\Phi}(2\mathcal{E}_r^+)}.$$

It is easy to see that for arbitrary points $x \in \mathcal{E}_r^+$ and $y \in (2\mathcal{E}_r^+)^c$ it holds

$$\frac{1}{2}\rho(x^0 - y) \le \rho(\widetilde{x} - y) \le \frac{3}{2}\rho(x^0 - y). \tag{3.5}$$

Then

$$\begin{aligned}
&\Big\|[b,\mathcal{R}]f_2(x)\Big\|_{L^{\Phi}(\mathcal{E}_r^+)} \lesssim \int_{(2\mathcal{E}_r^+)^c} \frac{|b(y)-b(x)|}{\rho(x^0-y)^{n+2}} |f(y)| dy \\
&\le \Big\| \int_{(2\mathcal{E}_r^+)^c} \frac{|b(y)-b_{\mathcal{E}_r^+}|}{\rho(x^0-y)^{n+2}} \, |f(y)| \, dy \Big\|_{L^{\Phi}(\mathcal{E}_r^+)} + \Big\| \int_{(2\mathcal{E}_r^+)^c} \frac{|b(x)-b_{\mathcal{E}_r^+}|}{\rho(x^0-y)^{n+2}} \, |f(y)| \, dy \Big\|_{L^{\Phi}(\mathcal{E}_r^+)} \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1 as follows

$$\begin{aligned}
I_1 &\lesssim \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{(2\mathcal{E}_r^+)^c} \frac{|b(y)-b_{\mathcal{E}_r^+}||f(y)|}{\rho(x^0-y)^{n+2}} \, dy \\
&= \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{(2\mathcal{E}_r^+)^c} |b(y)-b_{\mathcal{E}_r^+}||f(y)| \int_{\rho(x^0-y)}^{\infty} \frac{dt}{t^{n+3}} \, dy \\
&= \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \int_{2r \le \rho(x^0-y) \le t} |b(y)-b_{\mathcal{E}_r^+}||f(y)| dy \, \frac{dt}{t^{n+3}} \\
&\lesssim \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \int_{\mathcal{E}_t^+} |b(y)-b_{\mathcal{E}_r^+}||f(y)| dy \, \frac{dt}{t^{n+3}}.
\end{aligned}$$

Applying Hölder's inequality, Lemma 3.2 and (3.1), we get

$$\begin{aligned} I_1 \lesssim & \Big(\frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \int_{\mathcal{E}_t^+} |b(y) - b_{\mathcal{E}_t^+}| |f(y)| dy \, \frac{dt}{t^{n+3}} \\ & + \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} |b_{\mathcal{E}_r^+} - b_{\mathcal{E}_t^+}| \int_{\mathcal{E}_t^+} |f(y)| dy \, \frac{dt}{t^{n+3}} \Big) \\ \lesssim & \Big(\frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \Big\| b(\cdot) - b_{\mathcal{E}_t^+} \Big\|_{L^{\tilde{\Phi}}(\mathcal{E}_t^+)} \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \, \frac{dt}{t^{n+3}} \\ & + \frac{1}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} |b_{\mathcal{E}_r^+} - b_{\mathcal{E}_t^+}| \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \, \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t} \Big) \\ \lesssim & \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \, \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t}. \end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \Big\| b(\cdot) - b_{\mathcal{E}_r^+} \Big\|_{L^{\Phi}(\mathcal{E}_r^+)} \int_{(2\mathcal{E}_r^+)^c} \frac{|f(y)|}{\rho(x^0 - y)^{n+2}} dy.$$

By Lemma 3.2 and applying the Fubini theorem we get

$$\begin{aligned} I_2 & \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{(2\mathcal{E}_r^+)^c} \frac{|f(y)|}{\rho(x^0 - y)^{n+2}} dy \\ & \le \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{(2\mathcal{E}_r^+)^c} |f(y)| dy \int_{\rho(x^0-y)}^{\infty} \frac{dt}{t^{n+3}} \\ & \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \left(\int_{\mathcal{E}_t^+} |f(y)| dy \right) \frac{dt}{t^{n+3}}. \end{aligned}$$

Applying the Hölder's inequality (see, Lemma 2.6), we get

$$\begin{aligned} & \int_{(2\mathcal{E}_r^+)^c} \frac{|f(y)|}{\rho(x^0 - y)^{n+2}} dy \lesssim \int_{2r}^{\infty} \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \|1\|_{L^{\tilde{\Phi}}(\mathcal{E}_t^+)} \frac{dt}{t^{n+3}} \\ & = \int_{2r}^{\infty} \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \frac{1}{\widetilde{\Phi}^{-1}(|\mathcal{E}_t^+|^{-1})} \frac{dt}{t^{n+3}} \approx \int_{2r}^{\infty} \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t}. \end{aligned} \tag{3.6}$$

Direct calculations give

$$\|[b, \mathcal{R}] f_2\|_{L^{\Phi}(\mathcal{E}_r^+)} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \, \Phi^{-1}\big(t^{-n-2}\big) \, \frac{dt}{t} \tag{3.7}$$

and the last estimate holds for all $f \in L^{\Phi}(\mathbb{D}_+^{n+1})$ satisfying (3.3). Thus

$$\begin{aligned} \|\mathcal{R} f\|_{L^{\Phi}(\mathcal{E}_r^+)} & \lesssim \|b\|_* \, \|f\|_{L^{\Phi}(2\mathcal{E}_r^+)} \\ & + \frac{\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(\mathcal{E}_t^+)} \, \Phi^{-1}\big(t^{-n-2}\big) \, \frac{dt}{t}. \end{aligned} \tag{3.8}$$

On the other hand,

$$\|f\|_{L^\Phi(2\mathcal{E}_r)} = \frac{C}{\Phi^{-1}\big(r^{-n-2}\big)}\,\|f\|_{L^\Phi(2\mathcal{E}_r)}\int_{2r}^{\infty}\Phi^{-1}\big(t^{-n-2}\big)\,\frac{dt}{t}$$
$$\leq \frac{C}{\Phi^{-1}\big(r^{-n-2}\big)}\int_{2r}^{\infty}\|f\|_{L^\Phi(\mathcal{E}_t^+)}\,\Phi^{-1}\big(t^{-n-2}\big)\,\frac{dt}{t} \tag{3.9}$$

which unified with (3.8) gives (3.4).

Q.E.D.

For proving our main results, we need the following estimate.

Lemma 3.7. If $b \in L^1_{\rm loc}(\mathbb{D}^{n+1}_+)$ and $\mathcal{E}^+_0 := \mathcal{E}^+(x_0, r_0)$, then

$$|b(x) - b_{\mathcal{E}^+_0}| \leq C|b,\mathcal{R}|\chi_{\mathcal{E}^+_0}(x)$$

for every $x \in \mathcal{E}^+_0$, where $b_{\mathcal{E}^+_0} = \frac{1}{|\mathcal{E}^+_0|}\int_{\mathcal{E}^+_0} b(y)dy$.

Proof. If $x, y \in \mathcal{E}^+_0$, then $\rho(\widetilde{x} - y) \leq \rho(\widetilde{x} - x_0) + \rho(y - x_0) < 2r_0$. We get $Cr_0^{-n-2} \leq \rho(\widetilde{x} - y)^{-n-2}$. Therefore

$$|b,\mathcal{R}|\chi_{\mathcal{E}^+_0}(x) = \int_{\mathcal{E}^+_0}|b(x) - b(y)|\rho(\widetilde{x} - y)^{-n-2}dy \geq Cr_0^{-n-2}\int_{\mathcal{E}^+_0}|b(x) - b(y)|dy$$
$$\geq Cr_0^{-n-2}\left|\int_{\mathcal{E}^+_0}(b(x) - b(y))dy\right| = C|b(x) - b_{\mathcal{E}^+_0}|.$$

Q.E.D.

Theorem 3.8. Let Φ be a Young function with $\Phi \in \Delta_2 \bigcap \nabla_2$, $b \in BMO(\mathbb{D}^{n+1}_+)$ and $\varphi_1, \varphi_2 : \mathbb{D}^{n+1}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions satisfying (1.2). Then the commutator operator $[b,\mathcal{R}]$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ in $M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$ and

$$\|[b,\mathcal{R}]f\|_{M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)} \leq C\|b\|_*\,\|f\|_{M^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)} \tag{3.10}$$

with constants independent of f.

Proof. By Lemma 3.6 we have

$$\|[b,\mathcal{R}]f\|_{M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)} \leq C\|b\|_*\sup_{x^0,\,r>0}\varphi_2(x^0,r)^{-1}\int_r^{\infty}\Big(1 + \ln\frac{t}{r}\Big)\,\|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n-2}\big)\,\frac{dt}{t}.$$

Applying the Theorem 2.1 to the above integral with

$$w(r) = \Phi^{-1}\big(r^{-n-2}\big),\ v_2(x^0,r) = \varphi_2(x^0,r)^{-1}, \qquad v_1(x^0,r) = \varphi_1(x^0,r)^{-1}\,\Phi^{-1}\big(r^{-n-2}\big),$$

$$g(x^0,r) = \|f\|_{L^\Phi(\mathcal{E}^+(x^0,r))}, \qquad H^*_w g(x^0,r) = \int_r^{\infty}\Big(1 + \ln\frac{t}{r}\Big)\,\|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))}w(t)dt$$

where the condition (2.2) is equivalent to (1.2) we get

$$\|[b,\mathcal{R}]f\|_{M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})} \lesssim \|b\|_* \sup_{x\in\mathbb{D}_+^{n+1},\, r>0} \varphi_1(x^0,r)^{-1}\,\Phi^{-1}\big(r^{-n-2}\big)\,\|f\|_{L^{\Phi}(\mathcal{E}^+(x^0,r))}$$
$$= \|b\|_*\,\|f\|_{M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})}.$$

Q.E.D.

Proof of Theorem 1.1. The first part of the theorem follows from Lemma 3.6 and Theorem 2.11. We shall now prove the second part. Let $\mathcal{E}_0^+ = \mathcal{E}^+(x_0,r_0)$ and $x\in\mathcal{E}_0^+$. It is easy to see that $\mathcal{R}\chi_{\mathcal{E}_0^+}(x) = 1$ for every $x\in\mathcal{E}_0^+$. Therefore, by Lemmas 2.5 and 3.7

$$1 = \Phi^{-1}(w(\mathcal{E}_0^+)^{-1})\|\mathcal{R}\chi_{\mathcal{E}_0^+}\|_{L^{\Phi}(\mathcal{E}_0^+)} \le \varphi_2(\mathcal{E}_0^+)\|\mathcal{R}\chi_{\mathcal{E}_0^+}\|_{M^{\Phi,\varphi_2}}$$
$$\le C\varphi_2(\mathcal{E}_0^+)\|\chi_{\mathcal{E}_0^+}\|_{M^{\Phi,\varphi_1}} \le C\frac{\varphi_2(\mathcal{E}_0^+)}{\varphi_1(\mathcal{E}_0^+)}.$$

Since this is true for every $\mathcal{E}_0^+$, we are done.

The third statement of the theorem follows from the other statements of the theorem.

□

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